Optimal Execution of Portfolio Transactions

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Part 1

Basic models

Outline

- introduction
- market impact model
  - Price dynamics
- optimal trading strategy

Abstract

Consider the execution of portfolio transactions with the aim of minimizing a combination of risk and market impact

$$\min_x \big(E(x) + \lambda V(x)\big)$$

- Efficient frontier
- Risk/reward tradeoff

Parameters

$$x_k = X - \sum_{j=1}^{k} n_j = \sum_{j=k+1}^{N} n_j \quad k = 0, \ldots, N$$

- X : total shares to trade
- x_k : remaining inventory at time k
- τ : time interval (τ =T/N)
- $t_k = k\tau$ for k = 0, 1, …, N : total passed time
- $n_k = x_{k-1} - x_k$: trade size at time t
- $v_k = \frac{n_k}{\tau}$: velocity (shares per unit time)
- in this paper : sell side

Price Dynamics

$$S_k = S_{k-1} + \sigma\tau^{1/2}\xi_k - \tau g\left(\frac{n_k}{\tau}\right) = S_0 + \sigma\tau^{1/2}\sum_{j=1}^{k}\xi_j - \tau\sum_{j=1}^{k} g(v_j)$$

ξ_j: standard normal deviation ($\mu = 0, \sigma = 1$)

σ : volatility of the asset

$g(v)$: permanent impact function

(function of average rate v, assume no drift term)

theoretical price

With Temporary impact

$$\tilde{S}_k = S_{k-1} - h\left(\frac{n_k}{\tau}\right)$$

S_{k-1}: including its own permanent impact and deviation

$h(v)$: temporary impact function

(function of average rate v)

actual price

Trading trajectories

$$\tilde{S}_k = S_{k-1} - h\left(\frac{n_k}{\tau}\right)$$

$$S_k = S_0 + \sigma\tau^{1/2}\sum_{j=1}^{k}\xi_j - \tau\sum_{j=1}^{k} g(v_j)$$

- Capture(gain): $\sum_{k=1}^{N} n_k \tilde{S}_k = X S_0 + \sum_{k=1}^{N}\left(\sigma\tau^{1/2}\xi_k - \tau g\left(\frac{n_k}{\tau}\right)\right)x_k - \sum_{k=1}^{N} n_k h\left(\frac{n_k}{\tau}\right)$
- Trading cost: $X S_0 - \sum_{k=1}^{N} n_k \tilde{S}_k = -\sum_{k=1}^{N}\left(\sigma\tau^{1/2}\xi_k - \tau g\left(\frac{n_k}{\tau}\right)\right)x_k + \sum_{k=1}^{N} n_k h\left(\frac{n_k}{\tau}\right)$

Our goal is to minimize the trading cost

$$\min_x \left(E(x) + \lambda V(x)\right)$$

Minimize $X S_0 - \sum_{k=1}^{N} n_k \tilde{S}_k = - \sum_{k=1}^{N} \left(\sigma \tau^{1/2} \xi_k - \tau g\left(\frac{n_k}{\tau}\right) \right) x_k + \sum_{k=1}^{N} n_k h\left(\frac{n_k}{\tau}\right)$

- E(x) : expected cost of trading cost

$$E(x) = \sum_{k=1}^{N} \tau x_k \, g\left(\frac{n_k}{\tau}\right) + \sum_{k=1}^{N} n_k \, h\left(\frac{n_k}{\tau}\right)$$

- V(x) : variation of trading cost

$$V(x) = \sigma^2 \sum_{k=1}^{N} \tau x_k^2 .$$

- We will show that for each value of λ such that E(x)+λV(x) is minimal

Discuss : Linear impact

$$S_k = S_0 + \sigma\tau^{1/2}\sum_{j=1}^{k}\xi_j - \tau\sum_{j=1}^{k} g(v_j)$$

$$\tilde{S}_k = S_{k-1} - h\left(\frac{n_k}{\tau}\right)$$

- For permanent

$g(v) = \gamma v$, where γ has units ($/share)/(share/time)

- For temporary

$h(v) = \varepsilon \operatorname{sgn}(n_k) + \eta v$

, where units of ε are $/share, and η are ($/share)/(share/time)

linear market impact model

$$E(x) = \sum_{k=1}^{N} \tau x_k \, g\left(\frac{n_k}{\tau}\right) + \sum_{k=1}^{N} n_k \, h\left(\frac{n_k}{\tau}\right)$$

$$V(x) = \sigma^2 \sum_{k=1}^{N} \tau x_k^2$$

- rewrite

(1) $$E(x) = \tfrac{1}{2}\gamma X^2 + \epsilon \sum_{k=1}^{N} |n_k| + \frac{\tilde{\eta}}{\tau} \sum_{k=1}^{N} n_k^2$$, where $$\tilde{\eta} = \eta - \tfrac{1}{2}\gamma\tau$$

(2) $$V(x) = \sigma^2 \sum_{k=1}^{N} \tau x_k^2$$

Case : Minimum impact (minimum expected cost)

$$n_k = \frac{X}{N} \quad \text{and} \quad x_k = (N-k)\frac{X}{N}, \qquad k = 1, \ldots, N.$$

The reasons why we take $n_k = \frac{X}{N}$:

$\because$ expected value of linear trading cost is $E(x) = \frac{1}{2}\gamma X^2 + \epsilon \sum_{k=1}^{N} |n_k| + \frac{\tilde{\eta}}{\tau} \sum_{k=1}^{N} n_k^2$ and the only non constant variable is n_k^2

$\therefore n_k = \frac{X}{N}$ will get the minimum expected cost

naïve strategy

Case : Minimum Variance

$$n_1 = X, \quad n_2 = \cdots = n_N = 0, \qquad x_1 = \cdots = x_N = 0$$

The reason why we take n_1=X:

∵ variance of linear trading cost is $V(x) = \sigma^2 \sum_{k=1}^{N} \tau x_k^2$, where x_k is remaining quantity of order

∴ when n_1=X, x_k=0 from k=1 to k=N => V(x)=0

The utility function

$$\min_x \left(E(x) + \lambda V(x)\right)$$, where λ is risk aversion

the term ‘utility’ in this paper is the function above, to prevent ambiguity, here we use ‘negative utility’ to describe the combination of trading cost and risk.

optimal strategy

- $\min_x(E(x)+\lambda V(x))$ => differentiate negative utility

$$\frac{\partial U}{\partial x_j}$$

- we get optimal x_j, n_j:

$$x_j = \frac{\sinh\left(\kappa(T - t_j)\right)}{\sinh\left(\kappa T\right)} X$$

$$n_j = \frac{2\sinh\left(\frac{1}{2}\kappa\tau\right)}{\sinh\left(\kappa T\right)} \cosh\left(\kappa\left(T - t_{j-\frac{1}{2}}\right)\right) X$$

where $\tilde{k}^2 = \frac{\lambda\sigma^2}{\tilde{\eta}}$, and $\tilde{k}^2 = \frac{2}{\tau^2}(\cosh(k\tau) - 1)$

Part 2

implementation, Drift

Outline

- Recap
  - implementation of basic model
  - different time scale
  - sensitivity analysis
- value of information (Drift)
  - introduction
  - market impact model
    - Price dynamics
  - optimal trading strategy
  - implementation

implementation

- Japan Tobacco Inc., 2914.T
- 2022/06/24 – 2022/06/30(5 days)
- Only discuss ‘selling’
- Ignore overnight holding risk
- Step:
    1. Decide parameters
    2. Plot optimal trajectory & efficient frontier under different λ (risk averse coefficient)
    3. Check the strategy is optimal (by observing negative utility function)
    4. Price movement

different time scale : DAY / HOUR / MIN

Parameter

- T = total trade time.
- N = amount of trade interval.
- τ = length of trade interval(T/N).
- X = units that we want to completely liquidate before time T.
- ε = linear temporary market impact parameter(by paper, 1/2 bid ask spread)
- η = linear temporary market impact parameter(by paper, bid ask spread / 0.01 daily volume)
- γ = linear permanent market impact parameter(by paper, bid ask spread / 0.1 daily volume)
- σ = daily volatility with price scaling

DAY

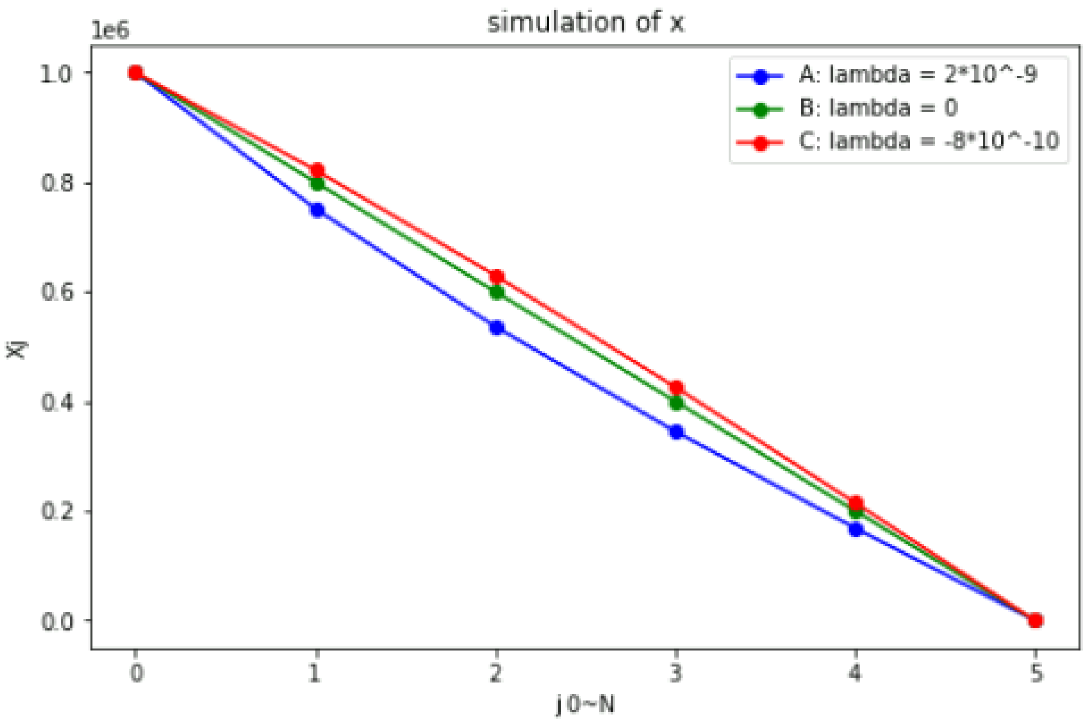
T = 5 ; N = 5

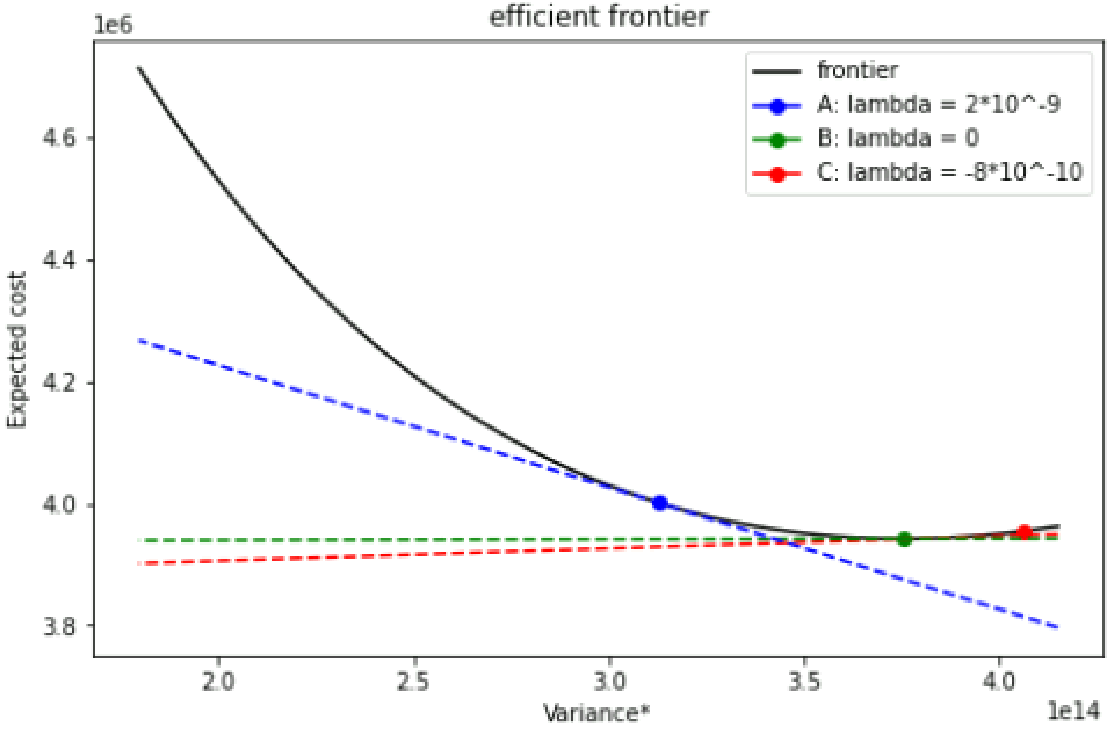
Parameter_DAY

- S_0=2430 (open price of 2022-06-24)
- T=5(days)
- N=5
- τ=1(days)
- X=10^6 shares
- ε(1/2 bid ask spread):0.374142
- η(bid ask spread / 0.01 daily volume) =1.4866*10^-5
- γ(bid ask spread / 0.1 daily volume)=1.4866*10^-6
- σ(daily volatility) = 17.6807

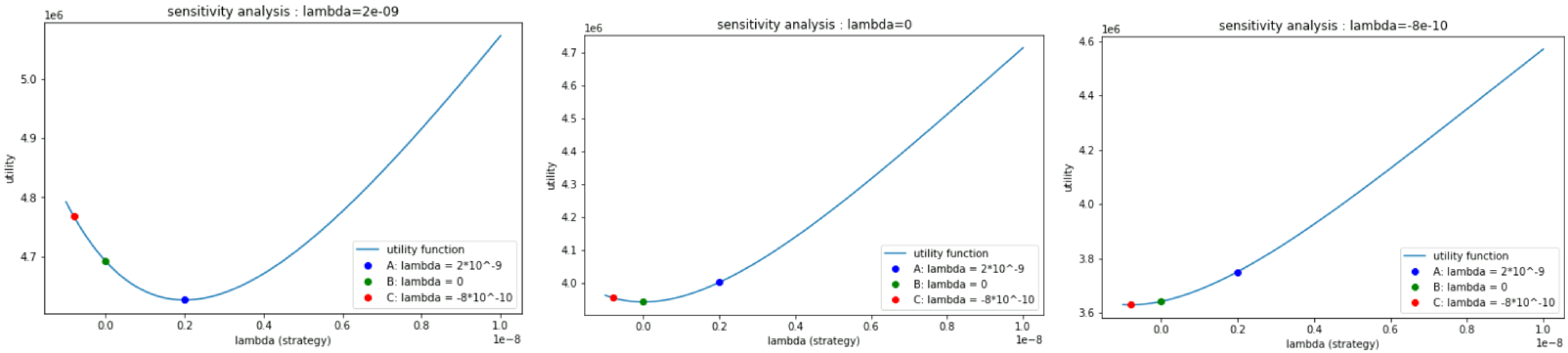
Optimal Trajectory & Efficient Frontier

- risk averse : λ = 2*10^-9
- risk neutral : λ =
- risk lover : λ = -8*10^-10





Sensitivity analysis



Comparison

+λV(x)

	1's day trade	2's day trade	3's day trade	4's day trade	5's day trade	Total volume (X)	Trading cost (realized trading cost)	Expected trading cost	Negative utility (Under λ=2*10^-9)	Negative utility (Under λ=0)	Negative utility (Under λ=-2*10^-10)
Risk averse	248465	215194	191449	176181	168711	10^6	41759718.42	4001201.92	5251553.31	4001201.92	3501061.37
Naïve strategy	200000	200000	200000	200000	200000	10^6	45870027.80	3941899.90	5317368.26	3941899.90	3391712.55
Risk Lovers	177884	192442	203593	211138	214944	10^6	47811304.69	3954702.42	5392762.10	3954702.42	3379478.55

- performance of diff λ (risk averse、naïve strategy、risk lover)
- Each strategy reaches optimal (minimum negative utility)
- Naïve strategy's expected trading cost is the smallest in all strategies

Price back testing

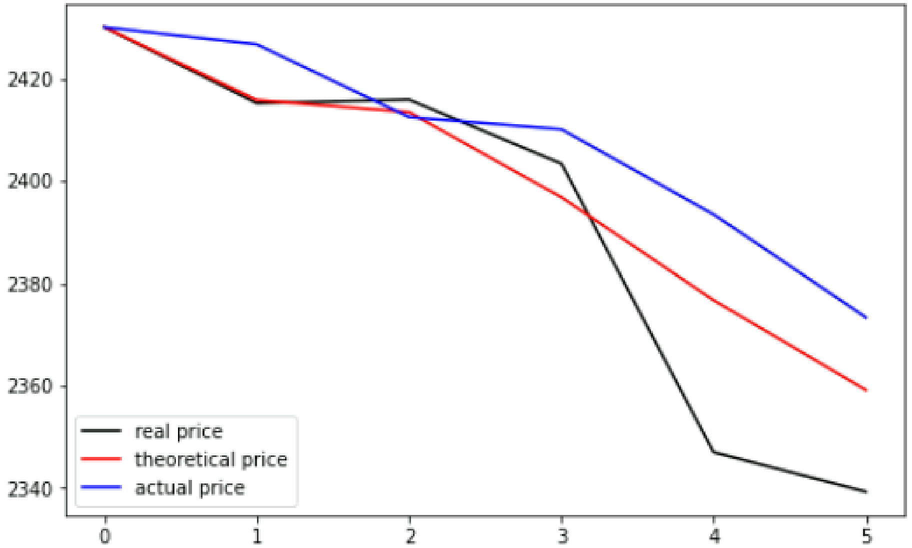
- Real price (by VWAP)
- Theoretical price (including permanent impact) :

  Fit ξ_j

$$S_k = S_0 + \sigma\tau^{1/2} \sum_{j=1}^{k} \xi_j - \tau \sum_{j=1}^{k} g(v_j)$$

- Actual Price (including permanent and temporary impact) :

$$\tilde{S}_k = S_{k-1} - h\left(\frac{n_k}{\tau}\right)$$



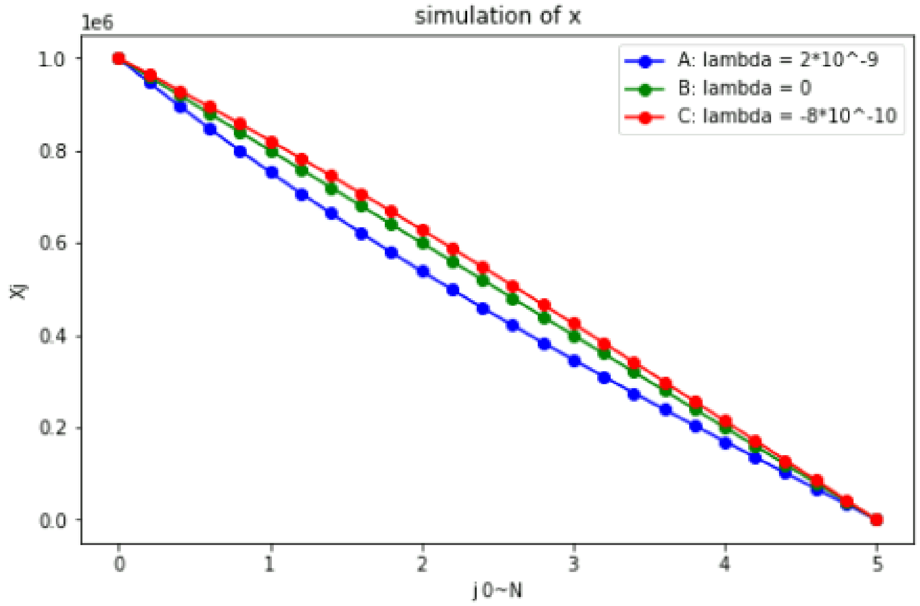
2914.T_price = [2430, 2415.18, 2415.85, 2403.35, 2347.00, 2339.27,]

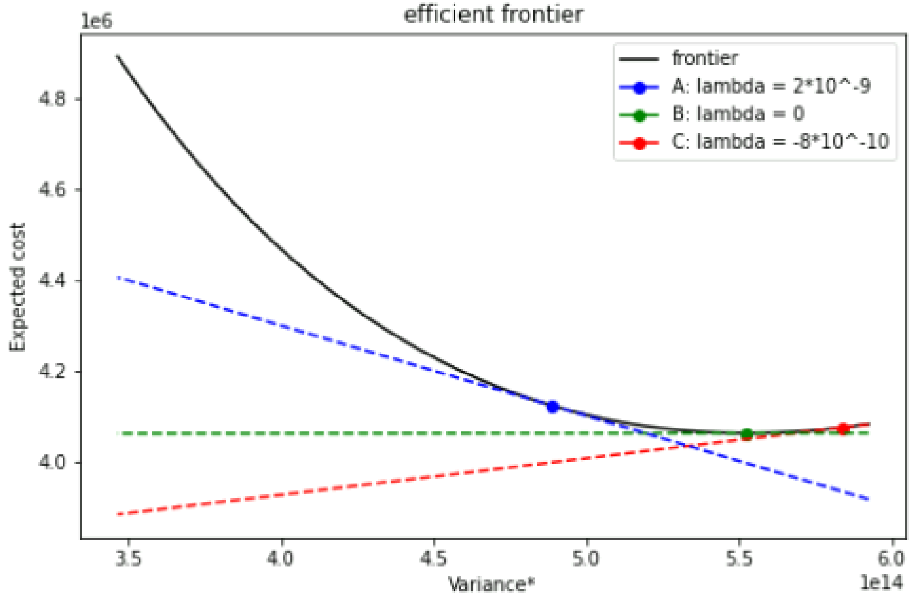
Hour

T = 5 ; N = 25

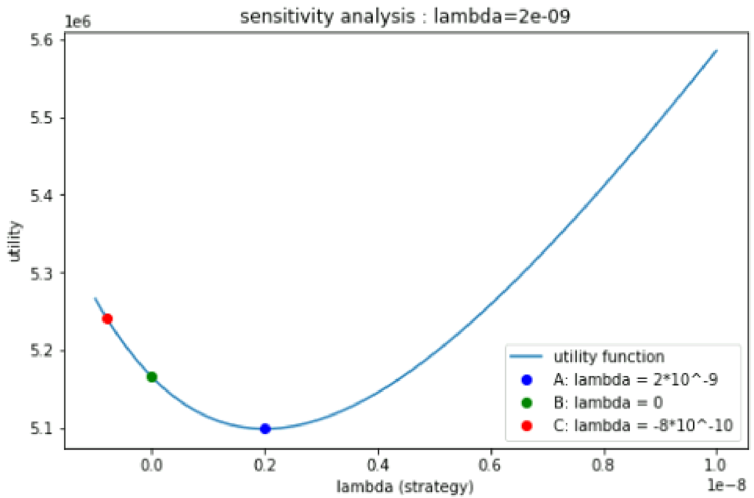
Optimal Trajectory & Efficient Frontier

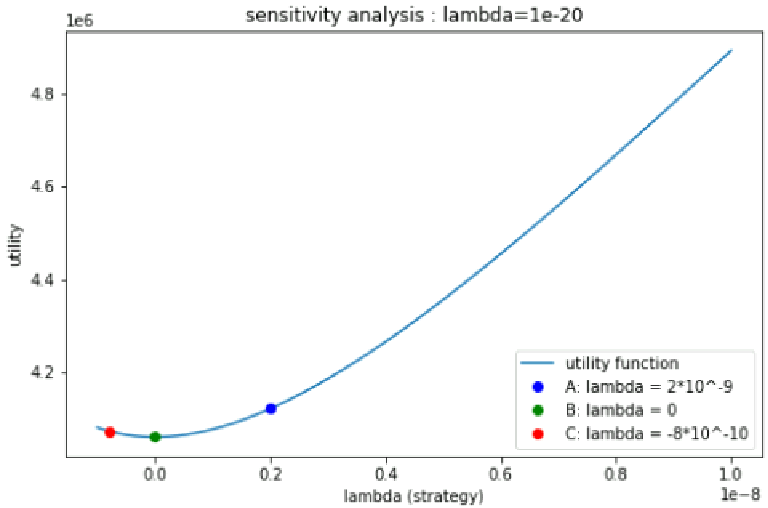
- risk averse : λ = 2*10^-9
- risk neutral : λ = 0
- risk lover : λ = -8*10^-10

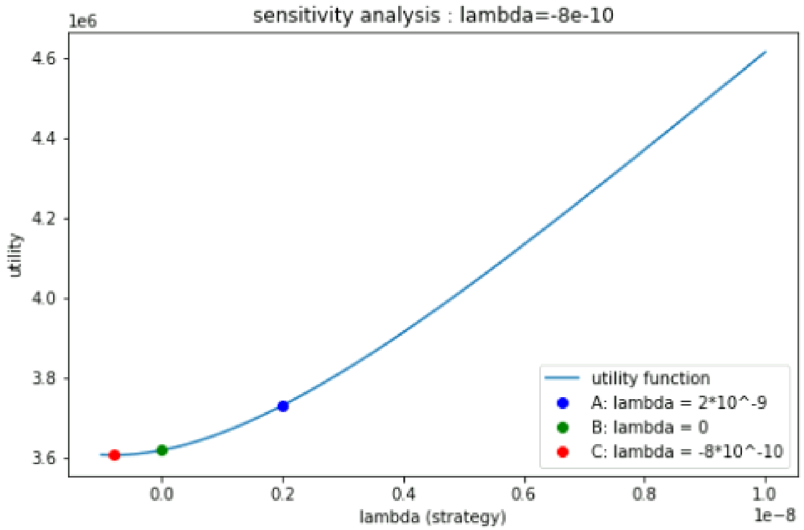




Sensitivity analysis







Comparison

+λV(x)

	1's day trade	2's day trade	3's day trade	4's day trade	5's day trade	Total volume (X)	Trading cost (realized trading cost)	Expected trading cost	Negative utility (Under λ=2*10^-9)	Negative utility (Under λ=0)	Negative utility (Under λ=-2*10^-10)
Risk averse	246816	214710	191756	176977	169741	10^6	41633125.27	4121612.13	5099315.81	4121612.13	3730530.65
Naïve strategy	200000	200000	200000	200000	200000	10^6	45573634.00	4060825.16	5166201.55	4060825.16	3618674.60
Risk Lovers	178842	192777	203441	210651	214288	10^6	47413449.39	4073626.09	5241704.58	4073626.09	3606394.70

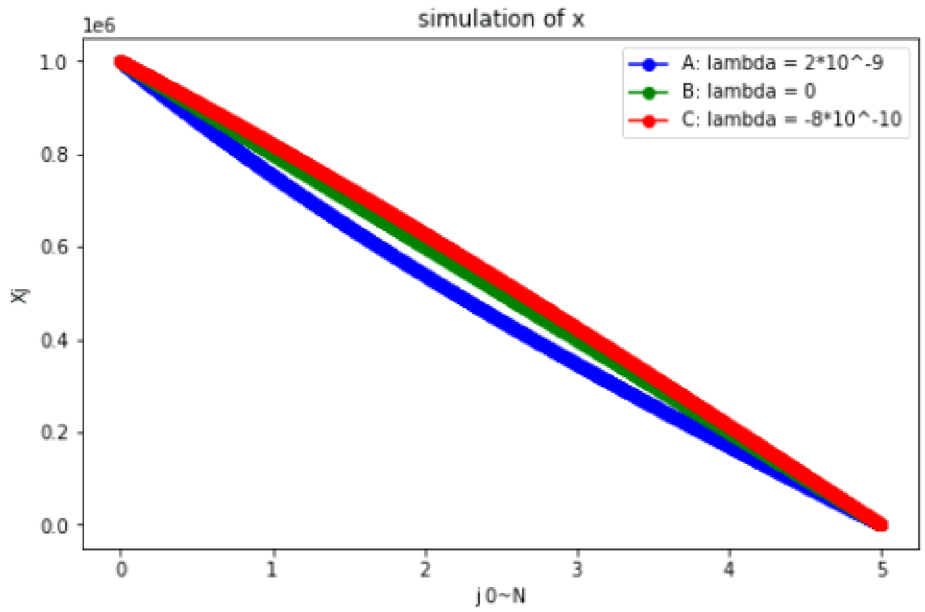
- performance of diff λ (risk averse、naïve strategy、risk lover)
- Each strategy reaches optimal (minimum negative utility)
- Naïve strategy's expected trading cost is the smallest in all strategies

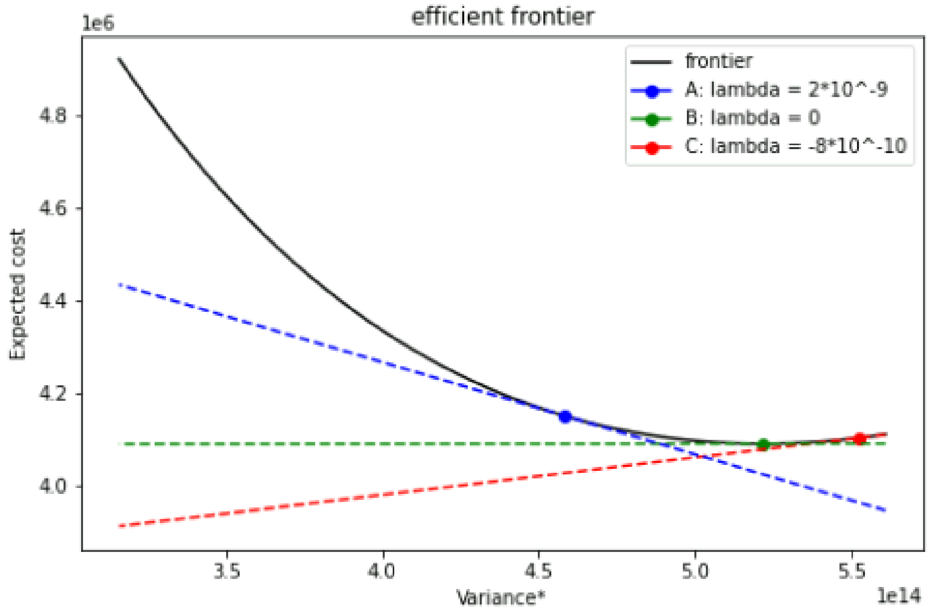
Minute

T = 5 ; N = 1500

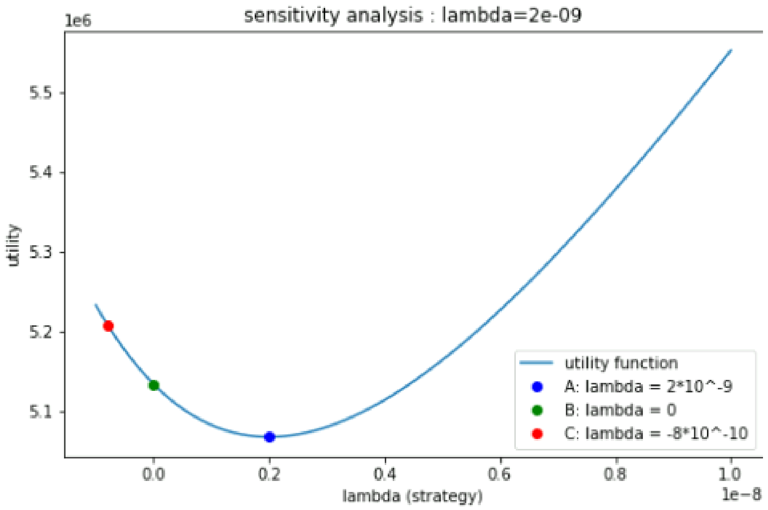
Optimal Trajectory & Efficient Frontier

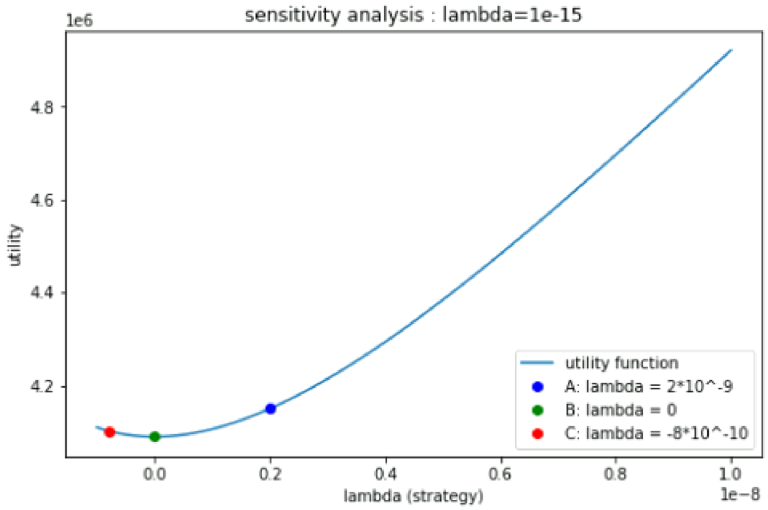
- risk averse : λ = 2*10^-9
- risk neutral : λ = 0
- risk lover : λ = -8*10^-10

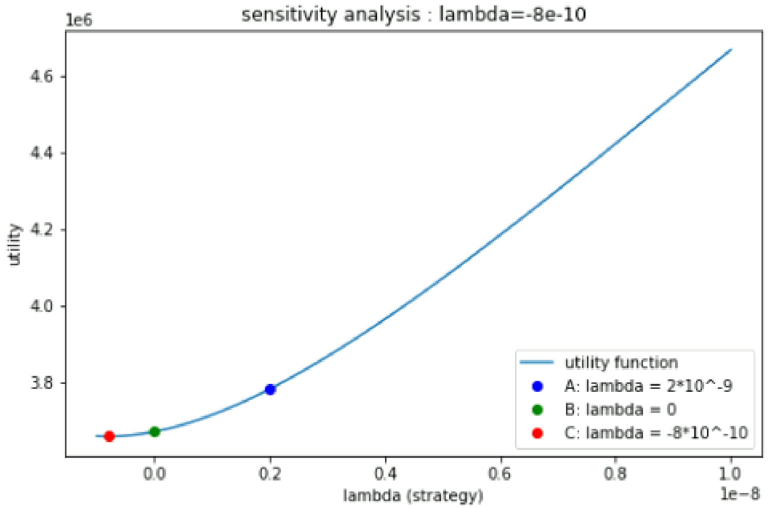




Sensitivity analysis







Comparison

$+\lambda V(x)$

	1's day trade	2's day trade	3's day trade	4's day trade	5's day trade	Total volume (X)	Trading cost (realized trading cost)	Expected trading cost	Negative utility (Under λ=2*10^-9)	Negative utility (Under λ=0)	Negative utility (Under λ=-2*10^-10)
Risk averse	246398	244587	191834	177179	170002	10^6	41511841.67	4150501.25	5066692.17	4150501.25	3784024.88
Naïve strategy	200000	200000	200000	200000	200000	10^6	45342480.65	4090060.95	5133124.62	4090060.95	3672835.49
Risk Lovers	179059	192853	200000	201542	214140	10^6	47129046.22	4102748.67	5207975.21	4102748.67	3660658.06

- performance of diff λ (risk averse、naïve strategy、risk lover)
- Each strategy reaches optimal (minimum negative utility)
- Naïve strategy's expected trading cost is the smallest in all strategies

Comparison (Day / Hour / Min)

以risk averse 為例 :

tradeoff

	1's day trade	2's day trade	3's day trade	4's day trade	5's day trade	Total volume (X)	Trading cost (realized trading cost)	Expected trading cost	Variance	Negative utility (Under λ=2*10^-9)
Day	248465	215194	191449	176181	168711	10^6	41759718.42	4001201.92	6.25*10^14	5251553.31
Hour	246816	214710	191756	176977	169741	10^6	41633125.27	4121612.13	4.89*10^14	5099315.81
Min	246398	244587	191834	177179	170002	10^6	41511841.67	4150501.25	4.58*10^14	5066692.17

- the shorter the time interval, the more accurate(detail) the trading strategy
- the shorter the time interval, the smaller the variance
- the shorter the time interval, the smaller the negative utility

Drift

Drift

- regard α as a drift parameter in a price process as a directional view of price movements:

$$S_k = S_{k-1} + \sigma\tau^{1/2}\xi_k + \alpha\tau - \tau g\left(\frac{n_k}{\tau}\right)$$

$$\tilde{S}_k = S_{k-1} - h\left(\frac{n_k}{\tau}\right)$$

- trading cost

$$X S_0 - \sum_{k=1}^{N} n_k \tilde{S}_k = -\sum_{k=1}^{N}\left(\sigma\tau^{1/2}\xi_k - \tau g\left(\frac{n_k}{\tau}\right)\right)x_k + \sum_{k=1}^{N} n_k h\left(\frac{n_k}{\tau}\right) - \sum_{k=1}^{N}(\alpha\tau)x_k$$

optimal strategy

- Recompute the expected costs and variance(with drift):

$$E(x) = \frac{1}{2}\gamma X^2 - \alpha \sum_{k=1}^{N} \tau x_k + \epsilon \sum_{k=1}^{N} |n_k| + \frac{\tilde{\eta}}{\tau} \sum_{k=1}^{N} n_k^2 \qquad V(x) = \sigma^2 \sum_{k=0}^{N-1} \tau x_k^2$$

- minimize negative utility :

zero-drift solution + correction

$$x_j = \frac{\sinh(\kappa(T - t_j))}{\sinh(\kappa T)} X + \left[1 - \frac{\sinh(\kappa(T - t_j)) + \sinh(\kappa t_j)}{\sinh(\kappa T)}\right] \bar{x}$$

$$n_j = \frac{2\sinh\left(\frac{1}{2}\kappa\tau\right)}{\sinh(\kappa T)} \cosh\left(\kappa\left(T - t_{j-\frac{1}{2}}\right)\right) X + \frac{2\sinh\left(\frac{1}{2}\kappa\tau\right)}{\sinh(\kappa T)} \left[\cosh\left(\kappa t_{j-\frac{1}{2}}\right) - \cosh\left(\kappa\left(T - t_{j-\frac{1}{2}}\right)\right)\right] \bar{x}$$

where $\bar{x} = \frac{\alpha}{2\lambda\sigma^2}$, the optimal level of security holding for a time-independent portfolio optimization problem

$$\bar{x} = \frac{\alpha}{2\lambda\sigma^2}$$

Problem - $\bar{x}$

- $\bar{x}$ is required to be smaller than X to make sure all the trades to be in same direction

$$0 \leq \bar{x} \leq X$$

- solution :
  - prompt warning message on console output in our program
  - redefine α

Parameter

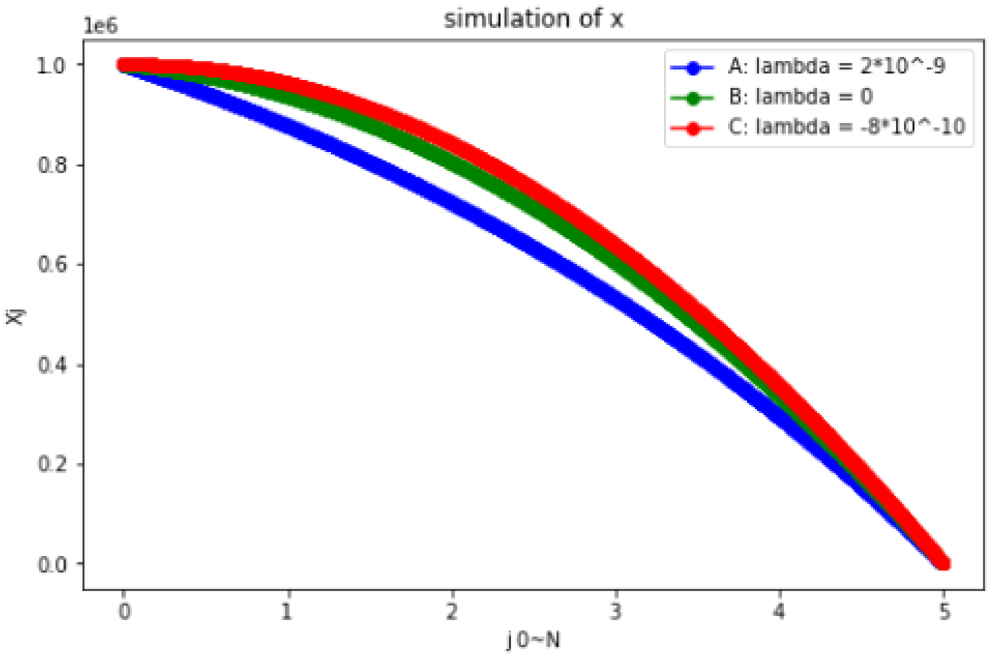
- T=5(days)
- N=1500
- τ=1(min)
- same X, ε, η, γ, σ
- α (by paper, CAGR with time scaling) = 2.026

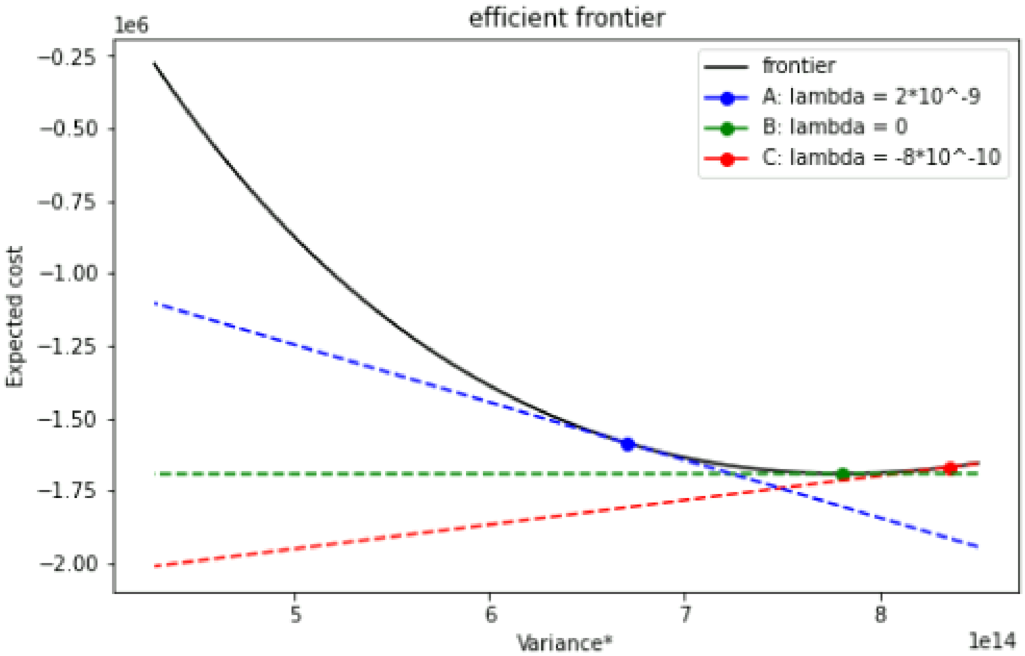
CAGR = $\frac{end\_price}{start\_price}^{\frac{1}{n}} - 1\ (compound\ annual\ growth\ rate)$

α = (CAGR/250)*S_0

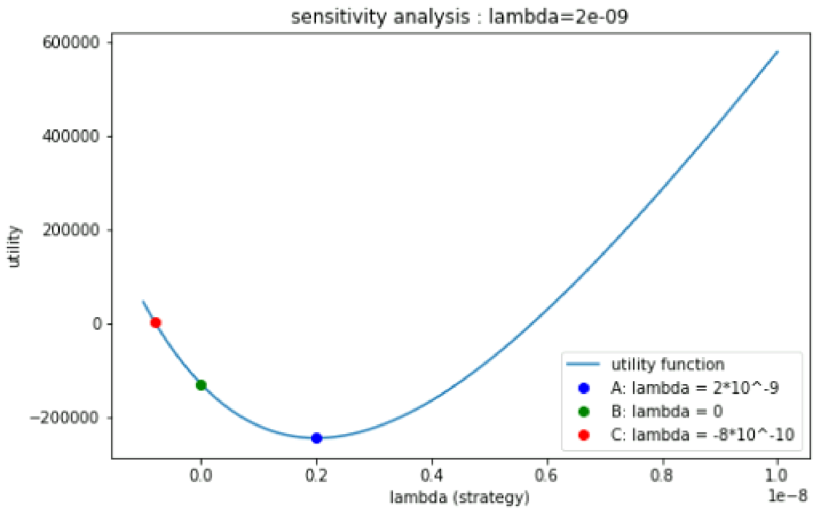
2914.T(minute with drift)

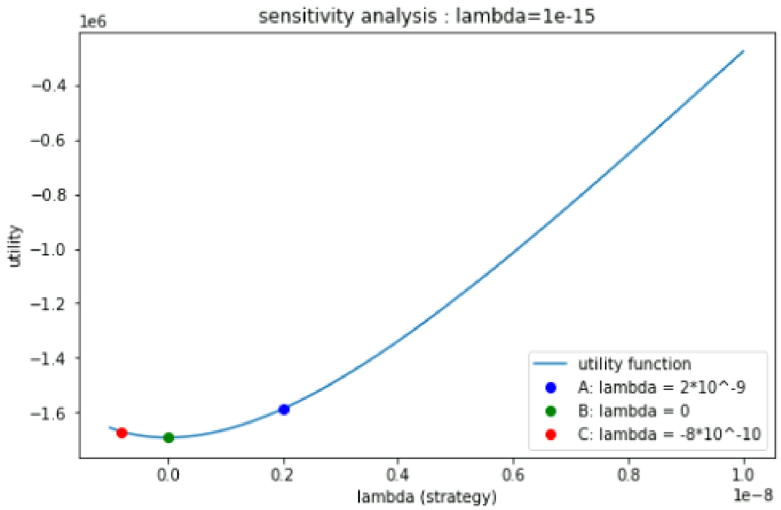
- risk averse : λ = 2*10^-9
- risk neutral : λ = 0
- risk lover : λ = -8*10^-10

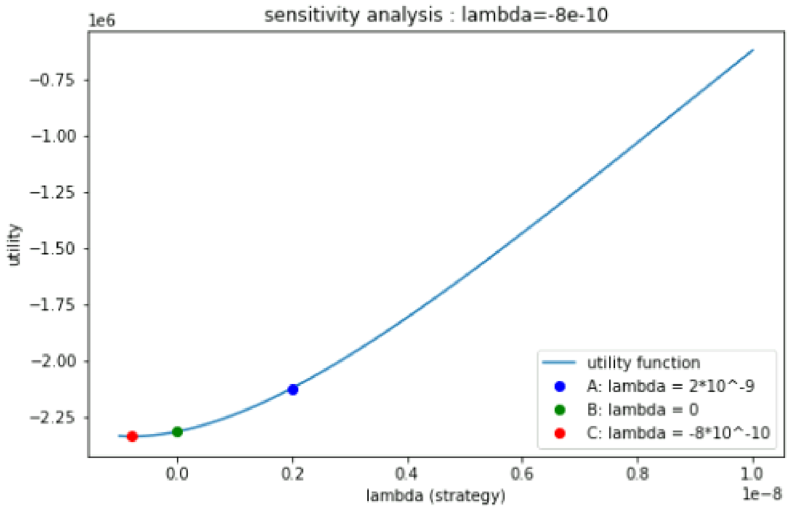




Sensitivity analysis







Comparison

+λV(x)

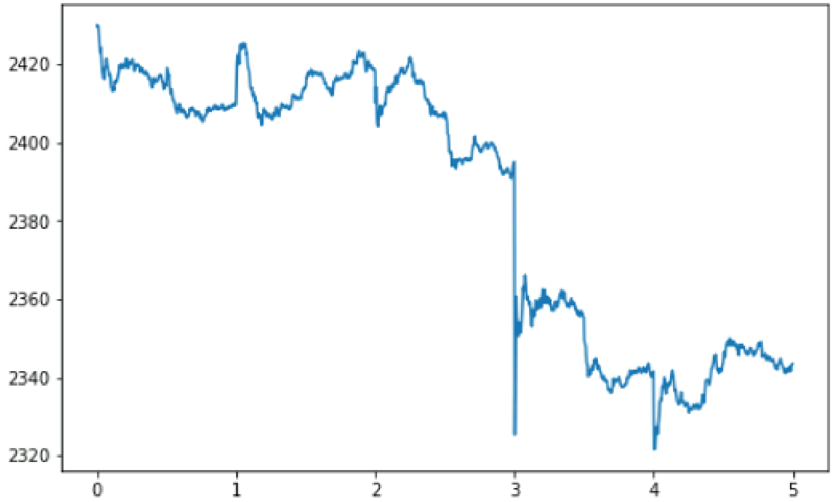
	1's day trade	2's day trade	3's day trade	4's day trade	5's day trade	Total volume (X)	Trading cost (realized trading cost)	Expected trading cost	Negative utility (Under λ=2*10^-9)	Negative utility (Under λ=0)	Negative utility (Under λ=-2*10^-10)
Risk averse	122594	153964	191834	237801	293807	10^6	54722764.22	-1586867.10	-245249.13	-1586867.10	-2123514.28
Naïve strategy	63722	131861	200000	268139	336278	10^6	59980694.79	-1692022.47	-129600.55	-1692022.47	-2316991.23
Risk Lovers	36933	121188	203406	282207	356266	10^6	62436580.03	-1669646.49	1950.37	-1669646.49	-2338285.24

- performance of diff λ (risk averse 、naïve strategy 、risk lover)
- Each strategy reaches optimal (minimum negative utility)
- Naïve strategy's expected trading cost is the smallest in all strategies
- expected cost is smaller than 0 because of α>0

Comparison (minute with drift / without drift)

	1's day trade	2's day trade	3's day trade	4's day trade	5's day trade	Total volume (X)	Trading cost (realized trading cost)	Expected trading cost	Variance	Negative utility (Under λ=2*10^-9)
With drift	122594	153964	191834	237801	293807	10^6	54722764.22	-1586867.10	6.71*10^14	-245249.13
Without drift	246398	244587	191834	177179	170002	10^6	41511841.67	4150501.25	4.58*10^14	5066692.17

α > 0 but



Conclusion

- with linear impact :
  - we can find optimal strategy by minimizing negative utility

$$\min_x(E(x)+\lambda V(x))$$

- considering value of information (drift)
- model may be extended in several ways :
  - continuous time τ -> 0
  - nonlinear impact function
  - time-varying coefficients

Ref.

- https://www.smallake.kr/wp-content/uploads/2016/03/optliq.pdf
- https://github.com/viai957/Optimal-Portfolio-Transactions/blob/master/Almgren%20and%20Chriss%20Model.ipynb
- https://www.imperial.ac.uk/media/imperial-college/research-centres-and-groups/cfm-imperial-institute-of-quantitative-finance/events/Lillo-Imperial-Lecture3.pdf